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A Math Classic: The Tale of Three Links

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Context

Investigation into linking properties of complete graphs has primarily been motivated by Drs. Conway and Gordon's seminal result:

Theorem 1. *[3] K_6 is intrinsically linked.*

In 2001, Drs. Flapan, Naimi and Pommerscheim published an article proving that K_{10} was *intrinsically triple linked* [1]. They also provided a counterexample proving K_9 was not intrinsically triple linked.

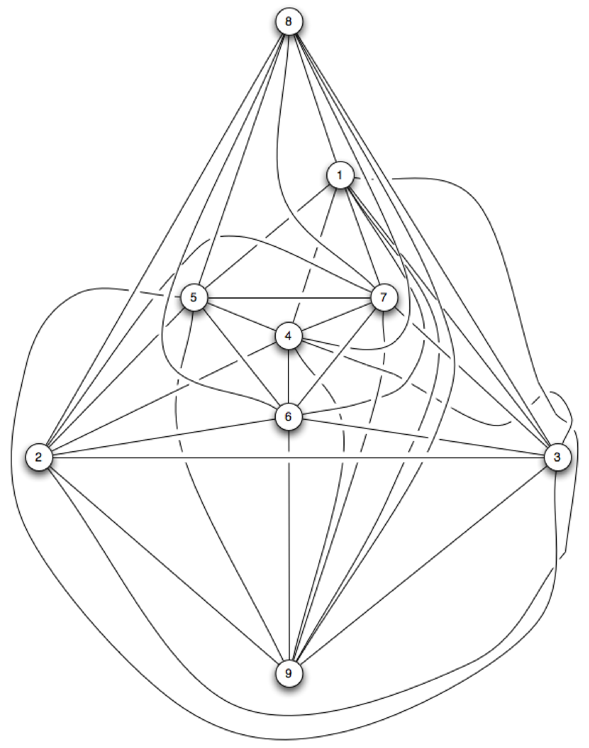

Figure 3: A *non-linear* embedding of K_9 that contains no triple link.

Terminology

Definition. A *complete graph* on n vertices, K_n is a graph in which all pairs of distinct vertices are connected by an edge. That is $\forall\, v_i, v_j \in V\ \exists\, (v_i, v_j) \in E$.

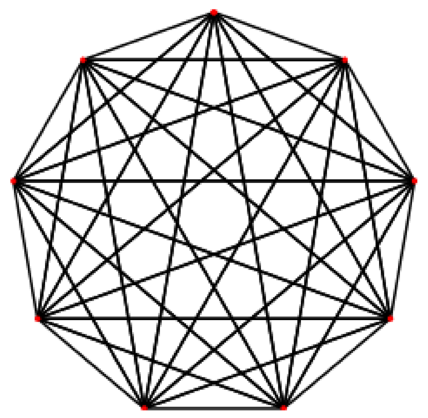

Figure 1: K_9, The complete graph on 9 vertices.

We disallow intersections of edges and instead, in a given embedding of the graph, force one edge "under" the other to form a *crossing*. This allows cycles within the graph to interact in interesting ways, i.e. *links*.

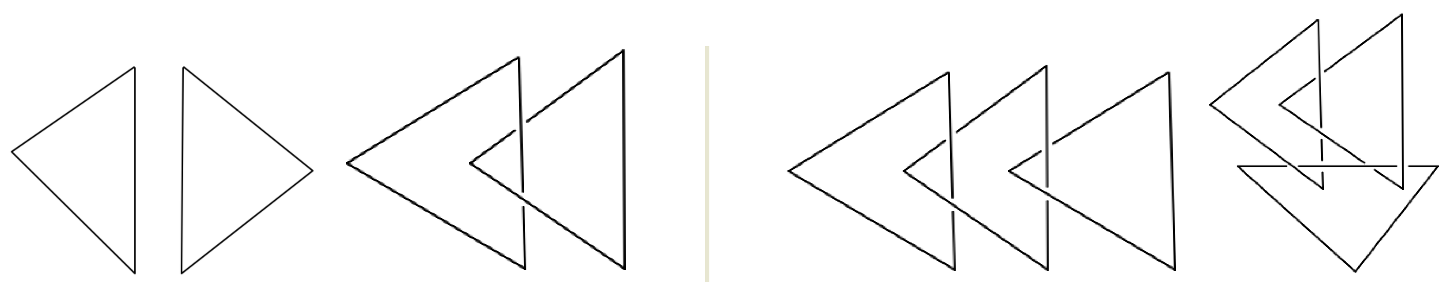

Figure 2: (Left) Two disjoint 3-cycles (triangles) and how they *link*. (Right) The two manifestations of *triple links*.

Are *linear* embeddings of K_9 intrinsically triple linked?

Linear embeddings of graphs are rather restrictive in their geometry, so how do we know there exists a linear embedding of K_9 with a triple link? Well there are groups of K_{10} graphs that we have been able to prove contain triple linked K_9 subgraphs. These depend on the concept of linked tetrahedra:

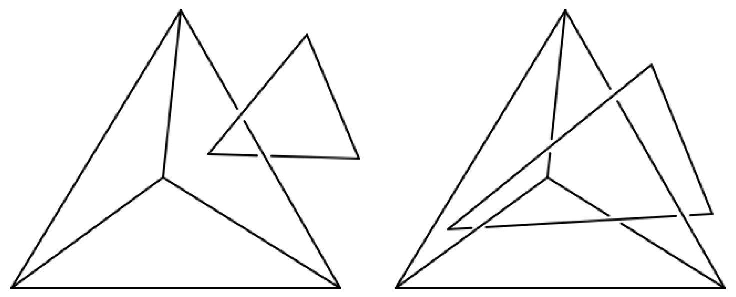

Figure 4: A 2-linked (left) and 4-linked (right) tetrahedron.

Proposition 1. *A linear embedding of K_{10} with a tetrahedron involved in a 4-link and 2-link, contains a triple-linked K_9 subgraph.*

Proposition 2. *A linear embedding of K_{10} with a tetrahedron involved in two adjacent 2-links contains a triple-linked K_9 subgraph.*

But now that we know there exist triple-linked linear K_9s, how do we prove whether or not they are all triple linked?

From K_{10} to K_9

As stated earlier, Drs. Flapan, Naimi and Pommerscheim were able to prove that K_{10} was intrinsically triple linked. We have been able to enhance their proof by narrowing our focus to linear embeddings.

Proposition 3. *Linear embeddings of K_{10} are intrinsically triple linked.*

In 2004, Drs. Bowlin and Foisy published an article discussing how Flapan's article might be improved [2].

Lemma. *[2] Let G be an embedded graph that contains a 4-pattern (respectively 6-pattern), as well as a vertex A, that is disjoint from the 4-pattern (6-pattern). Let S denote the vertices of the 4-pattern (6-pattern) minus the vertices of B, the triangle creating the pattern. If the vertices of $S \cup \{A\}$ induce a complete subgraph of G, then G contains a 3-link.*

Theorem 2. *[2] The graph obtained from K_{10} by removing four edges incident to a common vertex is intrinsically 3-linked.*

Theorem 3. *[2] The graph obtained from K_{10} by removing two nonadjacent edges is intrinsically 3-linked.*

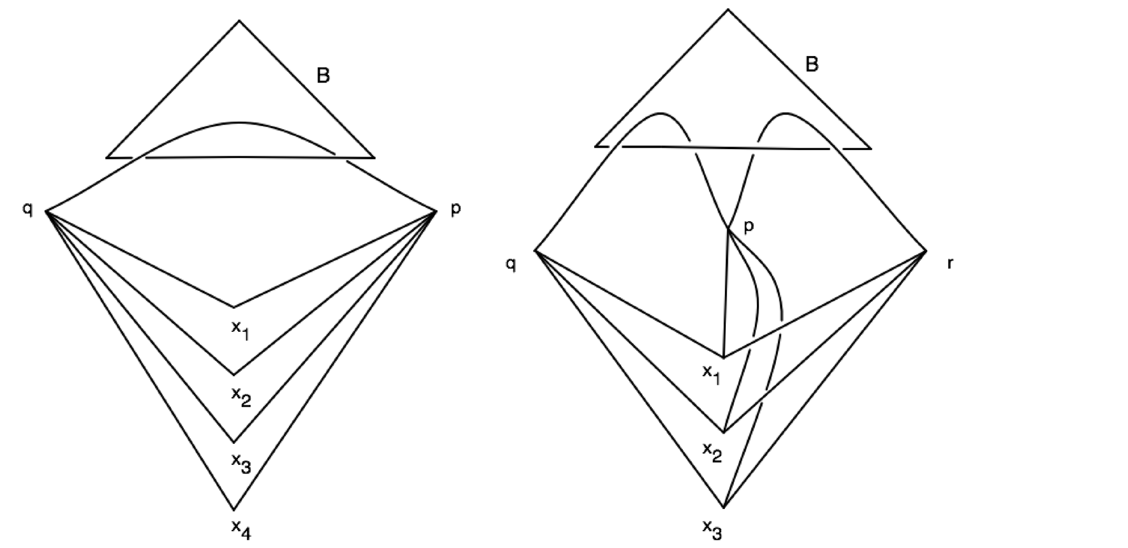

Figure 5: A 4-pattern (left) and 6-pattern (right).

We've been able to improve this somewhat:

Theorem 4. *The linear graph obtained from K_{10} by removing 7 edges incident to a common vertex is intrinsically 3-linked.*

Looking Ahead

Proposition 4. *[1] Every embedding of K_9 with no triple link has a triangle with a 6-pattern.*

Logically then, to be able to prove that every linear embedding of K_9 with a 6-pattern also has a triple link would prove that every linear embedding of K_9 had a triple link.

Proposition 5. *Every linear embedding of K_9 with a 6-pattern contains a triple link.*

Proof. Consider the 6-pattern presented in the previous section. B is linked with triangles $pqx_1, pqx_2, pqx_3, prx_1, prx_2$ and prx_3. Notice that without the vertices of B, we are left with 6 vertices, forming two tetrahedra that share an edge (WLOG $\overline{px_3}$). Consider these tetrahedra to be defined as follows:

$$C = pqx_1x_3$$

$$D = prx_2x_3$$

By definition then B forms either 2 or 4-linked tetrahedra with C and D. Consider the possible scenarios then:

- **Case 1:** B **forms 4-linked tetrahedra with both** C **and** D.
- **Case 2:** B **forms a 4-linked tetrahedron with** C **and a 2-link tetrahedron** D.
- **Case 3:** B **forms 2-linked tetrahedra with both** C **and** D.

□

This last case has been the only elusive portion of the proof so far, though insights from Hughes' [4] article are very promising.

References

[1] E. Flapan, R. Naimi and J. Pommersheim. *Intrinsically Triple Linked Complete Graphs.* Topology and its Applications 115 (2001), 239-246.

[2] G. Bowlin and J. Foisy *Some new intrinsically 3-linked graphs.* Journal of Knot Theory and its Ramifications 13(8) (2004), 10211027.

[3] J. Conway and C.McA. Gordon. *Knots and links in spatial graphs.* Journal of Graph Theory 7 (1983), 445-453.

[4] C. Hughes. *Linked Triangle Pairs in a Straight Edge Embedding of K6.* Pi Mu Epsilon Journal 12(4) (2006), 213-218.