

# Integer Construction by Induction

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## ABSTRACT

In 1889, Giuseppe Peano inductively defined the natural numbers by using the empty set along with a successor function. The natural numbers can be defined inductively primarily because they are well-ordered, a property which is equivalent to that of induction [1]. Inductive systems are especially useful in the area of computing for both reasoning about and implementing algorithms. Moreover, induction lends itself well to certain aspects of automated proving.

The natural numbers are just one example of inductive systems. Other inductive systems include string induction, tree induction, and transfinite induction [2, 3]. Although it would seem reasonable to describe sets containing the natural numbers inductively, such as the integers and rational numbers, traditional approaches have not done so. These systems have traditionally been defined as equivalence classes of natural numbers [4]. One reason may be that the integers and rationals are not well-ordered under the usual ordering. This leaves us with an intriguing question: Can the integers and rationals be described inductively? Here we present one possible well-ordering that allows us to define the integers inductively. By introducing several definitions, we are able to prove the common additive properties of the integers, including the associativity and commutativity of addition. This work motivates the future investigation of other systems, such as the rational numbers.

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## 1. INTRODUCTION

Induction is a powerful mathematical tool. The first known proof by mathematical induction can be found in Francesco Maurolico's *Arithmeticonum Libri Duo*. He used an inductive argument to prove that the sum of the first  $n$  odd natural numbers is  $n^2$  [5]. We now know that induction is valuable in many contexts, such as reasoning about recursion and algorithm implementation, as well as automated theorem proving. This leads us to a question of broad interest: What sets can be defined inductively?

Thanks to Peano, we already know that the natural numbers can be defined inductively. Peano declared that  $\mathbb{N}$ , the set of natural numbers,  $0$ , an element of  $\mathbb{N}$ , and the function  $\text{suc}: \mathbb{N} \rightarrow \mathbb{N}$ , must follow the following axioms, stated in slightly more modern terminology:

**Axiom 1:** There is no  $n \in \mathbb{N}$  such that  $\text{suc}(n) = 0$ .

**Axiom 2:** If  $n, m \in \mathbb{N}$ , and if  $\text{suc}(m) = \text{suc}(n)$ , then  $n = m$ .

**Axiom 3:** If  $A \subseteq \mathbb{N}$ ,  $0 \in A$ , and  $\text{suc}(n) \in A$  for every  $n \in A$ , then  $A = \mathbb{N}$ .

Axiom 1 states that  $0$  is not in the range of  $\text{suc}$ , Axiom 2 declares that  $\text{suc}$  is injective, and Axiom 3 is the inductive axiom. It should also be noted that when Peano first proposed this set of axioms in 1889, he used  $1$  as the least natural number (i.e., there is no  $n \in \mathbb{N}$  such that  $\text{suc}(n) = 1$ ), but it should be clear that this does not significantly affect the system being defined [1]. Since the set of integers contains the natural numbers, it would seem reasonable to use an inductive approach to introduce the integers. However, historically the integers have been defined by using equivalence classes of natural numbers. Each equivalence class contains all of the pairs of natural numbers with a common difference [4]. For example,  $(2, 1)$  and  $(4, 3)$  would both be in the same equivalence class.

Since the natural numbers are typically defined inductively, why not the integers? One of the reasons that the natural numbers can be defined inductively is that they are well-ordered, a property logically equivalent to induction. For a set  $S$  to be well-ordered under a given ordering,  $\leq$ , it means that any subset  $A$  of  $S$  contains a least element (i.e., there exists an  $a_0 \in A$  such that  $a_0 \leq a$  for all  $a \in A$ ). We note that the usual ordering on the integers is not a well ordering. For example, the set  $\{0, -2, -4, \dots\}$  does not have a least element. Therefore, a logical first step toward defining integers inductively is to find a well-ordering on the integers.

## 2. THE INTEGERS AS AN INDUCTIVE SYSTEM

Although there are infinitely many ways to order the integers, we chose the following ordering:

0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, ...

To differentiate this ordering from the usual one ( $\leq$ ), we choose the symbol  $\preceq$  to represent this new way to order integers. This new ordering is, in fact, a well-ordering, since every subset now has a "least" element. 0 now acts as the "least" integer, because  $0 \preceq z$  for all  $z \in \mathbb{Z}$ , where  $\preceq$  can be thought of as "is left of" with respect to the above list of integers. Motivated by Peano's axiom system for the natural numbers, in which a successor function is part of the description, we also define a function, NB, to suggest "next bounce." Essentially, NB allows us to move from integer to the integer on its immediate right with respect to the above well-ordering.

We are now ready to define three axioms of our own:

**Axiom 1:** There is no  $z$  in  $\mathbb{Z}$  such that  $NB(z) = 0$ .

**Axiom 2:** If  $x, y \in \mathbb{Z}$ , and if  $NB(x) = NB(y)$ , then  $x = y$ .

**Axiom 3:** If  $A \subseteq \mathbb{Z}$ ,  $0 \in A$ , and  $NB(z) \in A$  for every  $z \in A$ , then  $A = \mathbb{Z}$ .

Axiom 1 states that 0 is not in the range of NB, Axiom 2 declares that NB is injective, and Axiom 3 is the inductive axiom. Note that although our axioms seem very similar to those stated by Peano, it will quickly become apparent that our function NB acts very differently than Peano's successor function. For example, NB takes the  $n^{\text{th}}$  integer to the  $(n+1)^{\text{st}}$  integer, with respect to the above well-ordering, while Peano's successor function essentially maps  $n$  to  $n + 1$ . For example, intuitively,  $NB(0) = -1$  and  $NB(-1) = 1$ , while  $\text{suc}(0) = 1$  and  $\text{suc}(1) = 2$ , where both NB and  $\text{suc}$  continue along their respective well-orderings. Of course, when we start out, we have no

concept of -1, 1, or any other integers; only 0 is mentioned within our axioms. The other integers must be defined inductively using the NB function.

As usual in any mathematical development, the choice of the organization of definitions and theorems is somewhat subjective, depending on what the developers find reasonable. In this particular case, we know the end results we want to obtain, and knowing those results, such as associativity of addition, we have introduced definitions, lemmas, and theorems in a progressive order that allows us to get to those important results in a straight forward manner. Here we present a sampling of those results, including some definitions, lemmas and theorems. Space constraints prevent us from including all the proofs, so we show some typical ones.

## 3. DEFINITIONS AND THEOREMS

We first define a function that allows us to move from a given integer to the one on its immediate left under  $\preceq$ . This function is almost an inverse to the "next bounce" function NB. We will refer to this "previous bounce" function as PB. An exception occurs at 0, since there is no integer to the left of 0. For convenience to avoid having an undefined value for  $PB(0)$ , we simply define  $PB(0)$  to be 0. We define PB inductively:

**Inductive Definition** of PB:  $\mathbb{Z} \rightarrow \mathbb{Z}$

- (i)  $PB(0) = 0$ ,
- (ii)  $PB(NB(z)) = z$ ;

There is an immediate corollary establishing the inverse relationship between NB and PB with the 0 exception. Our proof uses the definition directly.

### Corollary 1

$\forall z \in \mathbb{Z}$  such that  $z \neq 0$ ,  $NB(PB(z)) = z$ .

### Proof:

Using the axioms previously given, we note that since  $z \neq 0$ ,  $z = NB(w)$  for some  $w \in \mathbb{Z}$ . Therefore,

$$\begin{aligned} NB(PB(z)) &= NB(PB(NB(w))) \\ &= NB(w), \text{ by def. of PB} \\ &= z \end{aligned} \quad \square$$

Besides the equivalence class approach to describing the integers, one could also construct the integers by taking the natural numbers, and including the additive inverse of each. That is, for each natural number  $n$ , a new number,  $-n$ , is defined such that  $n + (-n) = 0$ . This new number,  $-n$ , is called the additive inverse of  $n$ . Note that 0 is its own additive inverse. However, this construction of the integers turns out to be rather lengthy and tedious, as a

good deal of machinery needs to be developed for this approach [4]. As an alternative, we choose to define the concept of negation inductively:

**Inductive Definition** of  $\text{Is\_Neg}: \mathbb{Z} \rightarrow \{\text{True}, \text{False}\}$

- (i)  $\text{Is\_Neg}(0) = \text{False}$ ,
- (ii)  $\text{Is\_Neg}(\text{NB}(z)) = \neg \text{Is\_Neg}(z)$ ;

A natural extension of negation is unary minus, which we denote, as usual, by  $-z$ :

**Definition**

$$\forall z \in \mathbb{Z}, -z = \begin{cases} \text{NB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{PB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases};$$

**Corollary 2**

$\text{Is\_Neg}(-z)$  if and only if  $z \neq 0$  and  $\neg \text{Is\_Neg}(z)$ .

**Proof:** Follows directly from the definitions of  $\text{Is\_Neg}$  and unary minus.  $\square$

We now list several lemmas that relate the concept of negation with the functions  $\text{NB}$  and  $\text{PB}$ . These lemmas will be especially useful in the proofs of the theorems that follow.

**Lemma 1**

$$\forall z \in \mathbb{Z}, \text{NB}(-z) = \begin{cases} \text{NB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } n = 0 \\ z & \text{otherwise} \end{cases}.$$

**Proof:**

$$\begin{aligned} \text{NB}(-z) &= \begin{cases} \text{NB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } n = 0 \\ \text{NB}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of neg.} \\ &= \begin{cases} \text{NB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } n = 0 \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by Cor. 1} \end{aligned}$$

$\square$

**Lemma 2**

$$\forall z \in \mathbb{Z}, \text{PB}(-z) = \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{PB}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} \text{PB}(-z) &= \begin{cases} \text{PB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{PB}(0) & \text{if } z = 0 \\ \text{PB}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of neg.} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{PB}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of PB} \end{aligned}$$

$\square$

**Lemma 3**

$$\forall z \in \mathbb{Z}, -\text{NB}(z) = \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} -\text{NB}(z) &= \begin{cases} \text{PB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of neg.} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of PB} \end{aligned}$$

$\square$

**Lemma 4**

$$\forall z \in \mathbb{Z}, -\text{PB}(z) = \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} -\text{PB}(z) &= \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ -\text{PB}(0) & \text{if } z = 0 \\ \text{NB}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by def. of neg.} \\ &= \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ -0 & \text{if } z = 0 \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by def. of PB and Cor. 1} \end{aligned}$$

$$= \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of neg.}$$

$\square$

Motivated by what we know about integers, we recognize the importance of knowing the results of repeated negation. Our first theorem deals with this essential concept. Theorem 1, as well as several other theorems below will look familiar. Our intent is not to suggest that

these statements are new, but rather to show that they can be proven in an inductive description of the integers.

**Theorem 1**

$$\forall z \in \mathbb{Z}, -(-z) = z.$$

**Proof:**

$$\begin{aligned} -(-z) &= - \begin{cases} \text{NB}(z) & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of neg.} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by def. of neg.} \\ &= z \quad \square \end{aligned}$$

Since Lemma 4 gave an expression for  $-\text{PB}(z)$ , we can now use Theorem 1 to get a non-inductive definition of  $\text{PB}(z)$ . We will be able to use the following corollary to substitute for the term  $\text{PB}(z)$ , which will help us to prove some of the lemmas and theorems that follow.

**Corollary 3**

$$\forall z \in \mathbb{Z}, \text{PB}(z) = \begin{cases} -\text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ -z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:** Proof follows directly from Lemma 4 and Theorem 1.  $\square$

In anticipation of the need to define addition and subtraction, we give inductive definitions of both the predecessor and the successor functions. These functions will greatly facilitate the definitions that follow. Intuitively, if  $z$  is any given integer,  $\text{pred}(z) = z - 1$  and  $\text{suc}(z) = z + 1$ , where  $-$  and  $+$  are the usual integer subtraction and addition. Of course, at this stage, we are using  $+$  and  $-$  in a purely intuitive way, since we have not yet built the machinery for addition and subtraction in our inductive approach.

We begin with an inductive definition of the predecessor function, and then we list several helpful lemmas. Lemma 5 gives a non-inductive definition of  $\text{pred}(z)$ , and lemmas 6 and 7 relate  $\text{pred}$  and negation.

**Inductive Definition** of  $\text{pred}: \mathbb{Z} \rightarrow \mathbb{Z}$

- (i)  $\text{pred}(0) = \text{NB}(0)$ ,
- (ii)  $\text{pred}(\text{NB}(z)) = \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases};$

**Lemma 5**

$$\forall z \in \mathbb{Z}, \text{pred}(z) = \begin{cases} \text{NB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{PB}(\text{PB}(z)) & \text{otherwise} \end{cases}.$$

**Proof:**

$$\begin{aligned} \text{pred}(z) &= \text{pred}(\text{NB}(\text{PB}(z))) \\ &= \begin{cases} \text{NB}(\text{NB}(\text{NB}(\text{PB}(z)))) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{PB}(\text{PB}(z)) & \text{otherwise} \end{cases}, \\ &\quad \text{by def. of pred.} \\ &= \begin{cases} \text{NB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{PB}(\text{PB}(z)) & \text{otherwise} \end{cases}, \\ &\quad \text{by Cor. 1} \quad \square \end{aligned}$$

**Lemma 6**

$$\forall z \in \mathbb{Z}, \text{pred}(-z) = \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} \text{pred}(-z) &= \begin{cases} \text{pred}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{pred}(0) & \text{if } z = 0 \\ \text{pred}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by def. of neg.} \\ &= \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{pred}(\text{NB}(\text{PB}(\text{PB}(z)))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by def. of pred. and Cor. 1} \\ &= \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{NB}(\text{NB}(\text{NB}(\text{PB}(\text{PB}(z)))))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by def. of pred.} \\ &= \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{NB}(\text{NB}(\text{PB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Cor. 1} \\ &= \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(0) & \text{if } z = 0 \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Cor. 1} \quad \square \end{aligned}$$

**Lemma 7**

$$-\text{pred}(z) = \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(0)) & \text{if } z = 0 \\ \text{PB}(\text{PB}(\text{PB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} -\text{pred}(z) &= \begin{cases} -\text{NB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ -\text{NB}(0) & \text{if } z = 0 \\ -\text{PB}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Lemma 5} \\ &= \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(0)) & \text{if } z = 0 \\ \text{PB}(\text{PB}(\text{PB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Lemmas 3 and 4} \quad \square \end{aligned}$$

Now we will do the same for the successor function. We will first define the function inductively, and then list several lemmas, some of which give alternate forms. Lemma 9, in particular, gives an alternative definition of successor that uses the predecessor function.

**Inductive Definition** of  $\text{suc}: \mathbb{Z} \rightarrow \mathbb{Z}$ 

- (i)  $\text{suc}(0) = \text{NB}(\text{NB}(0))$ ,
- (ii)  $\text{suc}(\text{NB}(z)) = \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{PB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases};$

**Lemma 9**Alternative Inductive Definition of  $\text{suc}(z): \mathbb{Z} \rightarrow \mathbb{Z}$ 

- (i)  $\text{suc}(0) = \text{NB}(\text{NB}(0))$ ,
- (ii)  $\text{suc}(\text{NB}(z)) = \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{NB}(\text{pred}(z)) & \text{otherwise} \end{cases};$

**Proof:** Note that this lemma is similar to the Inductive Definition of  $\text{suc}(z)$  above. The only things that remain to prove are that  $\text{suc}(\text{NB}(0)) = 0$ , and that  $\text{NB}(\text{pred}(z)) = \text{PB}(z)$ , if  $\neg \text{Is\_Neg}(z)$ .

$$\begin{aligned} \text{suc}(\text{NB}(0)) &= \text{PB}(0), & \text{by def. of suc} \\ &= 0, & \text{by def. of PB} \end{aligned}$$

$$\begin{aligned} \text{If } \neg \text{Is\_Neg}(z), \text{NB}(\text{pred}(z)) &= \text{NB}(\text{PB}(\text{PB}(z))), \\ &\quad \text{by Lemma 5} \\ &= \text{PB}(z), \\ &\quad \text{by Cor. 1} \quad \square \end{aligned}$$

Lemma 10 gives a non-inductive definition of  $\text{suc}(z)$ :

**Lemma 10**

$$\forall z \in \mathbb{Z}, \text{suc}(z) = \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} \text{If } z = 0, \text{suc}(0) &= \text{NB}(\text{NB}(0)), \text{ by def. of suc.} \\ \text{If } z \neq 0, \text{suc}(z) &= \text{suc}(\text{NB}(\text{PB}(z))), \text{ by Cor. 1} \\ &= \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(\text{NB}(\text{PB}(z)))) & \text{if } \neg \text{Is\_Neg}(z) \text{ and } z \neq 0 \end{cases}, \\ &\quad \text{by def. of suc} \\ &= \begin{cases} \text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \text{ and } z \neq 0 \end{cases}, \\ &\quad \text{by Cor. 1} \quad \square \end{aligned}$$

Lemmas 11, 12, and 13 relate the successor function and negation.

**Lemma 11**

$$\forall z \in \mathbb{Z}, \text{suc}(-z) = \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(\text{NB}(z)) & \text{if } z = 0 \\ \text{PB}(\text{PB}(\text{PB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} \text{If } z = 0, \text{suc}(-0) &= \text{suc}(0), \text{ by def. of neg.} \\ &= \text{NB}(\text{NB}(0)), \text{ by def. of suc} \\ \text{If } z \neq 0, \text{suc}(-z) &= \begin{cases} \text{suc}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{suc}(\text{PB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \text{ and } z \neq 0 \end{cases}, \\ &\quad \text{by def. of neg.} \\ &= \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{suc}(\text{NB}(\text{PB}(\text{PB}(z)))) & \text{if } \neg \text{Is\_Neg}(z) \text{ and } z \neq 0 \end{cases}, \\ &\quad \text{by def. of suc and Cor. 1} \\ &= \begin{cases} \text{NB}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{PB}(\text{PB}(\text{PB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \text{ and } z \neq 0 \end{cases}, \\ &\quad \text{by def. of suc} \quad \square \end{aligned}$$

**Lemma 12**

$$\forall z \in \mathbb{Z}, -\text{suc}(z) = \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} -\text{suc}(z) &= \begin{cases} -\text{PB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ -\text{NB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Lemma 10} \end{aligned}$$

$$= \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases},$$

by Cor. 3 and Lemma 3

□

**Lemma 13**

$$\forall z \in \mathbb{Z}, -\text{suc}(z) = \begin{cases} \text{NB}(\text{suc}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

**Proof:**

$$\begin{aligned} -\text{suc}(z) &= \begin{cases} \text{PB}(z) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by Lemma 12} \\ &= \begin{cases} \text{NB}(\text{PB}(\text{PB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by Cor. 1} \\ &= \begin{cases} \text{NB}(\text{suc}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{NB}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \text{ by Lemma 10} \end{aligned}$$

□

The following corollary shows the relationship between the predecessor and successor functions:

**Corollary 4**

$$\forall z \in \mathbb{Z}, \text{pred}(-z) = -\text{suc}(z).$$

**Proof:** The proof follows directly from Lemmas 12 and 6. □

Theorem 2 exhibits the inverse relationship between pred and suc.

**Theorem 2**

$$\forall z \in \mathbb{Z}, \text{pred}(\text{suc}(z)) = z = \text{suc}(\text{pred}(z)).$$

**Proof:**

$$\begin{aligned} \text{pred}(\text{suc}(z)) &= \begin{cases} \text{pred}(\text{PB}(\text{PB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{pred}(\text{NB}(\text{NB}(z))) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Lemma 10} \\ &= \begin{cases} \text{pred}(-\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ \text{PB}(\text{NB}(z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Lemma 4 and def. of pred} \\ &= \begin{cases} \text{NB}(\text{PB}(z)) & \text{if } \text{Is\_Neg}(z) \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Lemma 6 and def. of pred} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ z & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\quad \text{by Cor. 1} \\ &= z \end{aligned}$$

$$\begin{aligned} \text{suc}(\text{pred}(z)) &= \begin{cases} \text{suc}(\text{NB}(\text{NB}(z))) & \text{if } \text{Is\_Neg}(z) \\ \text{suc}(\text{NB}(0)) & \text{if } z = 0 \\ \text{suc}(\text{PB}(\text{PB}(z))) & \text{otherwise} \end{cases}, \\ &\quad \text{by Lemma 5} \\ &= \begin{cases} \text{PB}(\text{NB}(z)) & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{suc}(\text{PB}(-z)) & \text{otherwise} \end{cases}, \\ &\quad \text{by def. of suc, and Lemmas 9 and 2} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{suc}(\text{NB}(\text{PB}(\text{PB}(-z)))) & \text{otherwise} \end{cases}, \\ &\quad \text{by def. of PB and Cor. 1} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{NB}(\text{NB}(\text{NB}(\text{PB}(\text{PB}(-z)))))) & \text{otherwise} \end{cases}, \\ &\quad \text{by def. of suc} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{NB}(\text{NB}(\text{PB}(-z))) & \text{otherwise} \end{cases}, \\ &\quad \text{by Cor. 1} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ \text{NB}(-z) & \text{otherwise} \end{cases}, \\ &\quad \text{by Cor. 1} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ 0 & \text{if } z = 0 \\ z & \text{otherwise} \end{cases}, \\ &\quad \text{by Lemma 1} \\ &= z \end{aligned}$$

□

We are now ready to define subtraction. Note that the symbol that we use to denote subtraction ( $-$ ) is slightly different from the symbol used to designate unary negation ( $\neg$ ).

**Inductive Def.** of subtraction: If  $m, n \in \mathbb{Z}$ , then

$$\begin{aligned} &\text{(i) } m - 0 = 0, \\ &\text{(ii) } m - \text{NB}(n) \\ &= \begin{cases} -(m - n) & \text{if } \text{Is\_Neg}(n) \\ \text{suc}(-(-m - n)) & \text{if } \neg \text{Is\_Neg}(n) \end{cases}; \end{aligned}$$

We now state some important lemmas concerning subtraction. We include a few typical proofs.

**Lemma 14**

$$\forall m, n \in \mathbb{Z}, -(m - n) = -m - (-n).$$

**Proof:**

$$\begin{aligned} \text{Base Case: } -(m - 0) &= -m, && \text{by def. of } - \\ &= -m - 0, && \text{by def. of } - \\ &= -m - (-0), && \text{by def. of neg.} \end{aligned}$$

Induction Hypothesis:  $-(m - n) = -m - (-n)$

$$\begin{aligned} -(m - \text{NB}(n)) &= \begin{cases} -(-(-m - n)) & \text{if } \text{Is\_Neg}(n) \\ -\text{suc}(-(-m - n)) & \text{if } \neg \text{Is\_Neg}(n) \end{cases}, \\ &\text{by def. of } - \\ &= \begin{cases} -(-(-m) - (-n)) & \text{if } \text{Is\_Neg}(n) \\ \text{suc}(-(-(-m - (-n)))) & \text{if } \neg \text{Is\_Neg}(n) \end{cases}, \\ &\text{by the Ind. Hyp.} \\ &= -m - (-\text{NB}(n)), \\ &\text{by def. of } - \end{aligned}$$

□

**Lemma 15**

$$\forall z \in \mathbb{Z}, 0 - z = -z.$$

**Proof:**

$$\begin{aligned} \text{Base Case: } 0 - 0 &= 0, && \text{by def. of } - \\ &= -0, && \text{by def. of neg.} \end{aligned}$$

Inductive Hypothesis:  $0 - z = -z$

$$\begin{aligned} 0 - \text{NB}(z) &= \begin{cases} -(-0 - z) & \text{if } \text{Is\_Neg}(z) \\ \text{suc}(-(-0 - z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\text{by def. of } - \\ &= \begin{cases} 0 - (-z) & \text{if } \text{Is\_Neg}(z) \\ \text{suc}(0 - (-z)) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\text{by Lemma 14 and def. of neg.} \\ &= \begin{cases} z & \text{if } \text{Is\_Neg}(z) \\ \text{suc}(z) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}, \\ &\text{by the Ind. Hyp.} \\ &= -\text{NB}(z), \\ &\text{by Lemmas 3 and 12} \end{aligned}$$

□

**Lemma 16**

$$\forall x, y \in \mathbb{Z}, \text{suc}(x - y) = x - \text{pred}(y).$$

We now define addition in terms of subtraction. We then state several theorems that relate subtraction and addition.

**Definition of Addition:**  $\forall m, n \in \mathbb{Z}, m + n = -(-m - n).$

The following lemma provides an inductive definition of addition in terms of subtraction and the successor function:

**Lemma 17**

$$m + \text{NB}(n) = \begin{cases} m - n & \text{if } \text{Is\_Neg}(z) \\ -\text{suc}(m - n) & \text{if } \neg \text{Is\_Neg}(z) \end{cases}.$$

Theorem 3 demonstrates that subtraction an integer  $n$  is equivalent to adding its inverse.

**Theorem 3**

$$\forall m, n \in \mathbb{Z}, m + (-n) = m - n.$$

**Proof**

$$\begin{aligned} m + (-n) &= -(-m - (-n)), && \text{by def. of } + \\ &= -(-m) - (-(-n)), && \text{by Lemma 14} \\ &= m - n, && \text{by Theorem 1} \end{aligned}$$

□

Theorem 4 shows that negations distributes over addition.

**Theorem 4**

$$\forall x, y \in \mathbb{Z}, -(x + y) = -x - y.$$

Theorem 5 gives alternate expressions for the successor and predecessor of a sum and difference.

**Theorem 5**

- (a)  $\forall m, n \in \mathbb{Z}, \text{suc}(m + n) = m + \text{suc}(n).$
- (b)  $\forall m, n \in \mathbb{Z}, \text{pred}(m + n) = m + \text{pred}(n).$
- (c)  $\forall m, n \in \mathbb{Z}, \text{suc}(m - n) = m - \text{pred}(n).$
- (d)  $\forall m, n \in \mathbb{Z}, \text{pred}(m - n) = m - \text{suc}(n).$

Finally, the last two theorems we will present verify that integer addition is commutative and associative under our inductive system. We withhold the proofs, for both are fairly long, but both theorems are proven by induction.

**Theorem 6**

$$\forall m, n \in \mathbb{Z}, m + n = n + m.$$

**Theorem 7**

$$\forall k, m, n \in \mathbb{Z}, (k + m) + n = k + (m + n).$$

**4. CONCLUSION**

We have presented an inductive approach to describing the set of integers with the operation of addition. As is necessary, the usual results of traditional integer theory hold. We plan to extend these methods to inductively define integer multiplication and to prove the usual properties. It will be interesting to discover what innovations result from this new way of thinking about the integers.

In fact, an interesting question concerns other systems that are countable. Since we took advantage of the countability of the integers to construct our well-ordering, it seems likely that our methods can be extended to other countable sets. The key to induction is well-ordering. Since every countable set can be well-ordered in some way (else, how would one determine it was countable), it should be possible to derive a function that allows for the generation of additional elements from the least element. Thus, at some point, the rational numbers may be described inductively, and will not require the previous construction of the natural numbers and the integers.

Perhaps most intriguing is the idea that *any* set can be well-ordered. In 1904, Ernst Zermelo proved that the Well-Ordering Theorem, which states that *any* set can be well-ordered, is logically equivalent to the Axiom of Choice [3]. Thus, if we accept the Axiom of Choice, we must also accept the possibility that any set can be described inductively. It is difficult to even begin to imagine an inductive definition of the real numbers. However, as we have demonstrated, a well-ordering is an important beginning to an inductive definition. Perhaps what is truly important is that the well-ordering itself be describable; existence may not be enough. In any case, we are inspired to carry out future investigation into the inductive properties of sets.

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